Hw3, 9.3, 9.5, 9.11, 9.14, 9.16a

9.3

In the near region the potentials are just like the static case with the time dependence $\exp(-i\omega t)$. The potential due to a sphere with oppposite voltages is given by equation 3.36 as

$$\Phi = \exp(-i\omega t)V\left(\frac{3a^2}{2r^2}P_1(\cos\theta) + \dots\right)$$

From this expansion we can read off the dipole term

$$\vec{p} = 6\pi\epsilon_0 V \exp(-i\omega t)a^2\hat{z}$$

Using 9.19 we find the fields in the radiation zone to be

$$H = -\frac{3\epsilon_0 Vca^2k^2\sin\theta}{2}\frac{\exp(ikr - i\omega t)}{r}\hat{\phi}$$
$$E = -\frac{3Va^2k^2\sin\theta}{2}\frac{\exp(ikr - i\omega t)}{r}\hat{\theta}$$

The time averaged angular power distribution is

$$\frac{dP}{d\Omega} = \frac{9V^2a^4}{8Z_0}\sin^2\theta\cos^2(\omega t),$$

which can be integrated to yield the total power

$$P = \frac{3\pi a^4k^4V^2}{Z_0}\cos^2(\omega t)$$

9.5

Keeping only the 1st term in 9.9, we have

$$A = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\int J(x')\,d^3x',$$

which can be manipulated as done in the book into

$$A = -\frac{i\mu_0\omega}{4\pi}\vec{p}\frac{\exp(ikr)}{r}.$$

Starting with the top equation on page 410, we can expand $\exp(ik|x - x'|)/|x - x'|$ to get an expression for the scalar potential very much like quation 9.6

$$\Phi = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\frac{\exp(ikr)}{r^{l+1}}\left(1 + a_1(ikr) + \dots\right)\int \rho(x')r'^l Y^*_{lm}(\theta',\phi')\,d^3x',$$

where the omitted terms come from the expansion of $j_l(kr')$. Keeping only the $l = 1$ term, the potential becomes

$$\Phi \approx \frac{1}{4\pi\epsilon_0}\frac{4\pi}{3}\frac{\exp(ikr)}{r^2}(1 - ikr)\sum_m Y_{lm}\int \rho(x')r'Y^*_{lm}(\theta',\phi')\,d^3x'$$

Using the addition theorem, equation 3.62, the RHS can be transformed into

$$\Phi \approx \frac{1}{4\pi\epsilon_0}\frac{\exp(ikr)}{r^2}(1-ikr)\hat{n}\cdot\vec{p}$$

From the given expressions for the potentials, we can use

$$H = \frac{1}{\mu_0}\bigtriangledown \times A$$
$$E = -\bigtriangledown \Phi - \frac{\partial A}{\partial t}$$

to find the fields. With the details omitted, they are simply

$$H = \frac{ck^2}{4\pi}\frac{\exp(ikr)}{r}\left(1+\frac{i}{kr}\right) n\times p$$
$$E = \frac{k^2}{4\pi\epsilon_0}\frac{\exp(ikr)}{r}(n\times p)\times n$$

9.11

This charge configuration, as we have learned from last quarter, is a quadrupole with components

$$Q_{11} = Q_{22} = 2qa^2\cos^2\omega t$$
$$Q_{33} = -2Q_{11}$$

By equation 9.45, we have

$$\frac{dP}{d\Omega} = \frac{Z_0 c^2 k^6 a^4 q^3}{32\pi^2}\cos^4\omega t \sin^2\theta\cos^2\theta,$$

which can be integrated to yield

$$P = \frac{Z_0 c^2 k^6 q^2 a^4}{60\pi}\cos^4\omega t$$

9.14

The current density is

$$J = \frac{I_0 \exp(i\omega t)}{2\pi a}\delta(r-a)\delta(\cos\theta)\hat{\phi}$$

In the radiation zone

$$A = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\int J(x')\exp(-ikn\cdot x')\, d^3x'$$
$$= \frac{\mu_0}{4\pi}\frac{a\exp(ikr)}{r}\int_0^{2\pi}\exp(-ika\sin\theta\cos\phi')\cos\phi'\, d\phi'$$
$$= \frac{\mu_0 a I}{2i}\frac{\exp(ikr)}{r}J_1(ak\sin\theta)\hat{\phi}$$

Because $H \sim \bigtriangledown \times A$, keeping only terms of $1/r$, we have

$$H = \frac{-Iak\exp(ikr)}{2ir} J_1(ak\sin\theta)\hat{\theta}$$

and

$$E = \frac{-Iak\exp(ikr)Z_0}{2r} J_1(ak\sin\theta)\hat{\phi}$$

Note that the magnetic and electric fields are out of phase and at 90 degrees to each other. It follows

$$\frac{dP}{d\Omega} = \frac{I^2a^2k^2Z_0J_1^2(ak\sin\theta)}{8}$$

The lowest nonvanishing moment is

$$M_{10} = \frac{Ia^2\sqrt{3\pi}}{2}$$

9.16

The current density is

$$\vec{J} = \hat{z}I\sin\left(\frac{2\pi z}{d}\right)$$

for $-d/2 \leq z \leq d/2$. By equation 9.8, we have

$$A = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\int J(x')\exp(-ikn\cdot x')\,d^3x',$$

which can be simplified to

$$\begin{aligned} A &= \hat{z}\frac{\mu_0 I\exp(ikr)}{4\pi r}\int_{-d/2}^{d/2}\sin(kz)\exp(-ikz'\cos\theta) \\ &= \hat{z}\frac{\mu_0 Id\exp(ikr)\sin(\pi\cos\theta)}{4\pi^2 r\sin^2\theta} \end{aligned}$$

Since

$$\begin{aligned} H &= ik\hat{n}\times A/\mu_0 \\ E &= ikZ_0(n\times A)\times n/\mu_0 \end{aligned}$$

the power distribution is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{2}Re[r^2 n\cdot E\times H^*] \\ &= \frac{1}{2}r^2\frac{k^2Z_0}{\mu_0^2}|n\times A|^2 \\ &= \frac{Z_0I^2\sin^2(\pi\cos\theta)}{8\pi^2\sin^2\theta} \end{aligned}$$

We see that there is no power parallel or perpendicular to the antenna. The maxima are at 45 degrees with it.