initial data. In particular, the observed dimensions of the fields (or other composite operators) will in general depend on the dynamics.

Of special interest is the case where the origin is an ultraviolet fixed point. This situation is called asymptotic freedom. Logarithmic corrections to naive scaling will then emerge as a result of renormalization.

13-2-1 Scale and Conformal Invariance

If in a classical action all dimensional constants vanish we would expect the theory to be scale invariant. This could also be the case in a massive theory at short distance (typically $m|x| \ll 1$).

If the configuration variables are scaled down

$$x \to {}^{\lambda}x = \lambda^{-1}x \qquad \lambda > 0 \tag{13-37}$$

the fields, noted generically as φ, would transform according to

$$\varphi(x) \to {}^{\lambda}\varphi(x) = U(\lambda)\varphi(\lambda x) \tag{13-37a}$$

with $U(\lambda)$ standing for a finite dimensional representation of the abelian group of dilatations. We shall assume this representation to be fully reducible. This means that we may write

$$U(\lambda) = e^{D \ln \lambda}$$

where the matrix D can be diagonalized. In infinitesimal form, when $\ln \lambda = \delta\varepsilon$, the transformation law reads

$$\delta\varphi = \delta\varepsilon\left(x \cdot \frac{\partial}{\partial x} + D\right)\varphi \tag{13-38}$$

In a classical massless theory (13-37) or (13-38) lead to an invariance provided the eigenvalues of D are equal to $1(d/2 - 1)$ for Bose fields, and $\frac{3}{2}(d/2 - \frac{1}{2})$ for Fermi fields. The quantities in parentheses apply to the case of an arbitrary dimension d instead of four.

We can also consider the effect of such transformations in a massive theory, obtaining therefore a Ward identity reflecting the breaking of scale invariance. In this sense we differ from pure dimensional analysis, since we consider the effect of a transformation of dynamical variables (the fields) but not of the dimensional parameters such as masses. If we were to do so we would relate two different physical situations.

For our favorite example‡

$$\mathscr{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - g\frac{\varphi^4}{4!} \tag{13-39}$$

‡ To avoid confusion with the scale parameter λ, the φ^4 coupling constant will be denoted g throughout this chapter.

we find a variation

$$\frac{\delta\mathscr{L}}{\delta\varepsilon} = x \cdot \frac{\partial\mathscr{L}}{\partial x} + 2(D+1)\frac{(\partial\varphi)^2}{2} - 4Dg\frac{\varphi^4}{4!} - 2D\frac{m^2}{2}\varphi^2 \tag{13-40}$$

Hence, if $D = 1$,

$$\frac{\delta\mathscr{L}}{\delta\varepsilon} = \left(x \cdot \frac{\partial}{\partial x} + 4\right)\mathscr{L} + m^2\varphi^2$$

The integral $\int d^4x\, \lambda^4\, \mathscr{L}(\lambda x)$ is independent of λ (positive). From a differentiation at $\lambda = 1$ it follows that

$$\int d^4x \left(x \cdot \frac{\partial}{\partial x} + 4\right)\mathscr{L}(x) = 0$$

meaning that $[x \cdot (\partial/\partial x) + 4]\mathscr{L}(x)$ is a divergence and that the variation of the action I reads

$$\delta I = \delta\varepsilon \int d^4x\, m^2\varphi^2(x) \tag{13-41}$$

Obviously, when m vanishes we have classical scale invariance.

Let us show that conformal invariance is then a consequence of scale invariance.

The conformal group is defined as the set of transformations leaving the angles invariant. This carries over to Minkowski space, where we deal with hyperbolic as well as circular angles. This group is obtained by adjoining to the Poincaré transformations an inversion with respect to an arbitrary point – the origin, for instance,

$$x^\mu \to x'^\mu = \frac{ax^\mu}{x^2} \tag{13-42}$$

For this definition to be meaningful, the usual R^4 space must be completed by a cone at infinity. Let us introduce a useful geometrical representation. We consider a six-dimensional space with a (2, 4) metric, i.e., such that

$$dz^2 = (dz_0)^2 - (dz_1)^2 - (dz_2)^2 - (dz_3)^2 - (dz_4)^2 + (dz_{-1})^2$$

The lines belonging to the isotropic cone $z^2 = 0$ are identified with the usual R^4 space completed by a cone at infinity. This may be realized, for instance, by cutting the cone $z^2 = 0$ by the hyperplane $z_{-1} = 1$ and projecting stereographically on R^4 the resulting one-sheeted hyperboloid from the point $(0, 0, 0, 0, -1)$ in R^5 (Fig. 13-3). The explicit formulas are

$$\begin{gathered} y_\mu = \frac{2x_\mu}{1 - x^2} \qquad y_4 = \frac{1 + x^2}{1 - x^2} \\ y_\mu y^\mu - (y_4)^2 = -1 \qquad 0 \le \mu \le 3 \\ x_\mu = \frac{y_\mu}{1 + y_4} \qquad x^2 = \frac{y_4 - 1}{y_4 + 1} \end{gathered} \tag{13-43}$$

The action of the four-sheeted pseudoorthogonal group $O(2, 4)$ is reflected on Minkowski space as conformal transformations. In particular dilatations correspond to hyperbolic rotations in the plane