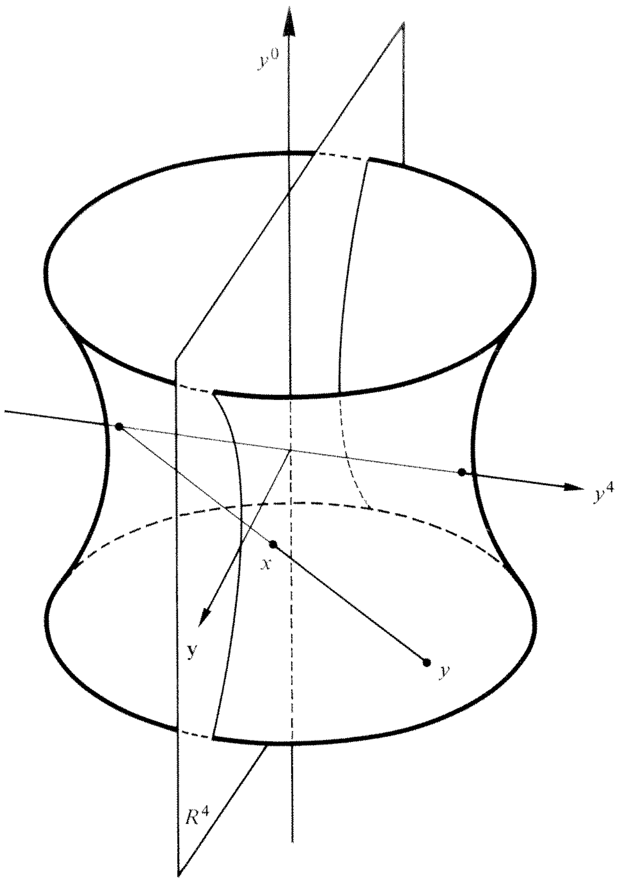

Figure 13-3 Projection of a (1, 4) hyperboloid on Minkowski space.

z_4, z_{-1}:

$$x' = e^{\theta}x \Leftrightarrow y'_\mu = \frac{y_\mu}{\cosh\theta + y_4\sinh\theta}, \; y'_4 = \frac{y_4\cosh\theta + \sinh\theta}{y_4\sinh\theta + \cosh\theta}$$

$$\Leftrightarrow z'_\mu = z_\mu, \quad z'_4 = z_4\cosh\theta + z_{-1}\sinh\theta, \quad z'_{-1} = z_{-1}\cosh\theta + z_4\sinh\theta \tag{13-44}$$

As an exercise, find the four other types of conformal transformations, completing therefore the number of generators to 15. Write the corresponding transformations in Minkowski space. Perform the similar construction in the case of an euclidean four-dimensional space. The conformal group is then $O(1, 5)$ and R^4, completed by a point at infinity, may be identified with the stereographic projection of a unit sphere in five-dimensional space.

In order to prove the conformal invariance of the massless φ^4 theory it is therefore sufficient to study the effect of an inversion. In five-dimensional y space this transformation corresponds to a symmetry $y \leftrightarrow -y$ of the unit hyperboloid. We have to choose a transformation law for the field. Equation (13-37a), ${}^{\lambda}\varphi(x) = \lambda\varphi(\lambda x)$, suggests the definition

$$\varphi(x) \to \varphi'(x) = \frac{1}{x^2}\varphi\left(\frac{x}{x^2}\right) \tag{13-45}$$

It follows that

$$I' = \int d^4x \left[\frac{1}{2}(\partial\varphi')^2 - g\frac{\varphi'^4}{4!}\right] = I + \int d^4x \frac{4}{x^2}\varphi(x)\left(1 + x\cdot\frac{\partial}{\partial x}\right)\varphi(x)$$

The additional term is a four-divergence

$$\frac{4}{x^2}\varphi\left(1 + x\cdot\frac{\partial}{\partial x}\right)\varphi = 2\partial_\mu\left[\frac{x^\mu}{x^2}\varphi^2(x)\right]$$

Formally (i.e., barring possible singularities) the action, and therefore the equations of motion, are conformally invariant.

Exercises

(a) Express the massless φ^4 theory in five-dimensional space with dynamical variables defined on a unit hyperboloid (minkowskian case) or unit sphere (euclidean case). Write the corresponding lagrangian and equations of motion. Expand the solutions of the classical free-field equations in terms of generalized spherical harmonics.

(b) Show that the variation of the action of a massive theory under a dilatation [Eq. (13-41)] may be written as the integral of the four-divergence of a dilatation current. The latter is related to a modified energy momentum tensor (such that its trace vanishes in the massless case) as follows:

$$\delta I = \int d^4x\,\partial_\mu S^\mu\,\delta\varepsilon$$

$$S^\mu = x_\nu T^{\nu\mu} \tag{13-46}$$

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathscr{L} + \tfrac{1}{6}(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\varphi^2$$

For a discussion see the work of Callan, Coleman, and Jackiw.

(c) Investigate scale and conformal invariance in the presence of Fermi fields.

13-2-2 Modified Ward Identities

We perform a Wick rotation and study the euclidean theory. For our problem of short-distance behavior this implies no restriction. We write the normalized generating functional as

$$e^{G(j)} = \int \mathscr{D}(\varphi)\exp\left\{-I + \int d^4x\,j\varphi\right\} \tag{13-47}$$

The euclidean action I may be split into

$$I = I_1 + I_2 \qquad I_1 = \int d^4x\left[\frac{1}{2}(\partial\varphi)^2 + g\frac{\varphi^4}{4!}\right] \qquad I_2 = \int d^4x\,\frac{m^2}{2}\varphi^2 \tag{13-48}$$

A scale transformation $\varphi(x) \to {}^{\lambda}\varphi(x)$ may be considered here as a simple change of integration variable, under which

$$I_1 \to I_1 \qquad I_2 \to \lambda^{-2}I_2 \qquad \int d^4x\,j\varphi \to \lambda\int d^4x\,j(x)\varphi(\lambda x)$$

The change in the measure is absorbed into the normalization and we recover