

LSZ reduction formula

1 Introduction

I will give the main ideas for justifying the LSZ reduction formula, which states how to obtain momentum-space S-matrix elements from off-shell Green functions in a quantum field theory.

The complication that bedevils any reasonable derivation is that the aim is to obtain a formula for S-matrix elements in momentum space. However, momentum eigenstates are spread out uniformly over the whole of space, and since they are eigenstates of the hamiltonian, they do *not* correspond to a situation that evolves nontrivially with time.¹ So momentum eigenstates do not really represent a system that is initially two incoming particles and that evolves to a system of outgoing particles. In addition, momentum eigenstates are not normalizable, so they are not genuine quantum mechanical states; they are merely convenient mathematical tools.

It is *compulsory* that a real derivation uses wave packet states, even though there is no mention of wave packets in the final LSZ reduction formula.

The structure of these notes is:

1. State the theorem.
2. Express genuine matrix elements for scattering in terms of a momentum basis.
3. Show how to create the states with fields acting on the vacuum in a simple way.
4. Use asymptotic properties at large positive and negative times to obtain the reduction formula.

The derivation is presented for a theory with a single relativistic scalar field. The formulation of generalizations for other fields, including non-relativistic fields, is relatively straightforward.

2 The reduction formula

Let $G_N(p_1, p_2, \dots, p_{N-1})$ be a *connected* N -point Green function, and let $\Gamma_N(p_1, p_2, \dots, p_{N-1})$ be the corresponding *amputated* Green function. It is convenient to choose them to be defined without the momentum conservation delta function. So there are only $N - 1$ momentum arguments, and the last momentum obeys $p_N = -\sum_{j=1}^{N-1} p_j$. We will also choose the convention that all the momentum are in-flowing.

Then the unamputated and amputated Green functions are related by

$$G_N(p_1, p_2, \dots, p_{N-1}) = \prod_{j=1}^N G_2(p_j) \Gamma_N(p_1, p_2, \dots, p_{N-1}). \quad (1)$$

¹The last part of this sentence is phrased so that it can be applied in the Heisenberg picture and so that the evolution by a phase e^{-iEt} in Schrödinger picture is ignored.

Let c be the square root of the residue of the pole of the 2-point function, aside from the standard factor of i :

$$G_2(p) = \frac{-ic^2}{p^2 + m_{\text{phys}}^2 - i\epsilon} + \text{non-pole term.} \quad (2)$$

The coefficient c is the normalization of the vacuum-to-one-particle matrix element:

$$\langle \mathbf{p} | \phi(x) | 0 \rangle = c e^{-ip \cdot x}, \quad (3)$$

with $p_0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m_{\text{phys}}^2}$. We may choose the phase factor of the normalization of the states so that c is real and positive.

The LSZ theorem [1] is that the connected part of the S-matrix element for $2 \rightarrow n$ scattering is given by

$$S_{\mathbf{q}_1, \dots, \mathbf{q}_n; \mathbf{p}_1, \mathbf{p}_2}^{\text{conn}} = (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{j=1}^n q_j \right) c^{n+2} \Gamma_{n+2}(p_1, p_2, -q_1, \dots, -q_{n-1}). \quad (4)$$

The external momenta of the Green function are set on shell. Therefore the prescription is

1. Replace each full external propagator of the full Green function by the factor c .
2. Set the external momenta on-shell.

In terms of the full Green function, this can be written as

$$S_{\mathbf{q}_1, \dots, \mathbf{q}_n; \mathbf{p}_1, \mathbf{p}_2}^{\text{conn}} = (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{j=1}^n q_j \right) \times \\ \times \lim_{\text{on-shell}} \frac{1}{c^{n+2}} \prod_{j=1}^2 \frac{p_j^2 + m_{\text{phys}}^2}{-i} \prod_{j=1}^n \frac{q_j^2 + m_{\text{phys}}^2}{-i} G_{n+2}(p_1, p_2, -q_1, \dots, -q_{n-1}). \quad (5)$$

Observe that the *full* Green function diverges when the external momenta are put on-shell. So, to get a valid construction, we must first multiply the Green function by the factors of $p_j^2 - m_{\text{phys}}^2$, etc, and then take the limit of on-shell momenta.

3 Matrix elements with wave packets

We let $|i\rangle$ and $|f\rangle$ be states² that have simple expressions in terms of the momentum-basis for in- and out-states:

$$|i\rangle = \sum_{\mathbf{p}_1, \mathbf{p}_2} \tilde{\psi}_1(\mathbf{p}_1) \tilde{\psi}_2(\mathbf{p}_2) |\mathbf{p}_1, \mathbf{p}_2; \text{in}\rangle, \quad (6)$$

$$|f\rangle = \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \tilde{\psi}'_1(\mathbf{q}_1) \dots \tilde{\psi}'_n(\mathbf{q}_n) |\mathbf{q}_1, \dots, \mathbf{q}_n; \text{out}\rangle. \quad (7)$$

Here the sum symbols over momenta represents the usual Lorentz-invariant integral over momenta:

$$\sum_{\mathbf{p}} \dots = \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} \dots \quad (8)$$

²Here ‘ i ’ is for ‘initial state’ and ‘ f ’ is for ‘final state’.

The $\tilde{\psi}$ s are single-particle momentum-space wave functions. (I use the $\tilde{}$ to represent momentum-space objects.) Unprimed $\tilde{\psi}$ s are for the initial state, while primed $\tilde{\psi}$ s are for the final state.

The state $|i\rangle$ is therefore a state which at very large negative times approaches a state of free separated individual particles with wave function $\tilde{\psi}_1(\mathbf{p}_1)\tilde{\psi}_2(\mathbf{p}_2)$. (To deal with the Bose symmetry of the particles, this should be symmetrized over p_1 and p_2 , a process which is taken care of by the symmetry of the basis states $|\mathbf{p}_1, \mathbf{p}_2; \text{in}\rangle$.)

Similar statements apply to the state $|f\rangle$, except that they are applied in the far future. Exactly how far one has to go to the past and future to have a good free-particle approximation depends on the states, i.e., on the wave functions. We could have considered more complicated non-product wave functions, but the case considered is sufficient.

Our strategy will be

- to obtain a formula for the overlap of the two states in terms of coordinate-space Green functions.
- to express this in terms of momentum-space Green functions.
- to convert this to the LSZ formula.

For each momentum-space wave function we define the corresponding coordinate-space wave function, e.g.,

$$\psi_1(x) = \sum_{\mathbf{p}} \tilde{\psi}_1(\mathbf{p}) e^{ip \cdot x}, \quad (9)$$

with p on shell, of course. Although, in general, we do not like using wave functions in relativistic theories, the concept of a wave function is valid in a *free*-field theory. Our use of wave functions is to label the content of a state in the infinite past or future when a state corresponds to (a linear combination) of isolated particle states. That is, we use the concept of wave function when the state is being correctly treated as a collection of free particles. However, the nature of these particles is determined by the interactions; for example in a Schrödinger field theory of electron and proton fields, every bound state energy level of hydrogen is a separate particle. (The easiest way to know that you have to treat these as separate particles in the strict context of scattering theory is that you can make a beam of atoms in any one energy level.)

We will assume that each wave function ψ_1 and ψ_2 corresponds to a wave packet state. Thus they correspond to propagation of free particles with approximately definite momenta. We will want to consider the following situation

- Very large negative times, so that the two particles correspond to widely separated incoming particles.
- Very narrow wave functions in momentum, so that the two particles have rather definite momenta.

If the momentum is almost definite, the position is very uncertain, so that we will have to take sufficiently large negative times to ensure that the two particles are physically widely separated.

4 Creation of states $|i\rangle$ and $|f\rangle$ by fields

The aim of this section is to find a formula for a state such as $|i\rangle$ as an integral of some kind of coordinate space wave function with fields applied to the vacuum:

$$|i\rangle = \iint d^4x_1 d^4x_2 \phi(x_1) \phi(x_2) |0\rangle g(x_1, x_2). \quad (10)$$

If you remember the corresponding formula in Schrödinger field theory, you might expect it to be sufficient to do an integral over the spatial coordinates only, at a fixed value of time. However this doesn't work so well in a relativistic field theory:

- Even in free-field theory, the fields have both destruction and creation operators in them. So applying two or more field operators to the vacuum will not create a state of definite particle number.
- In free-field theory, we can construct an expression involving both fields and the canonical momentum fields at a fixed time. But this will not help us in the interacting case, because of the next item:
- In an interacting theory, the fields create add not only single particles but multiple particles.³

The last point can be understood intuitively in terms of an energy-time uncertainty relation: if we apply a field operator at a definite time, the energy of the state can be infinitely uncertain. In a relativistic theory a large uncertainty in energy means that there is sufficient energy to create extra particles. Although the energy-time uncertainty relation is not as conceptually clean as the momentum-space uncertainty principle in elementary quantum mechanics, the idea is intuitively reasonable. Of course, it may be possible to choose fields very carefully so that they only create one particle.

However it is much simpler to arrange that to use an integral over time with a suitable time-dependent wave function that selects the desired energy of a particle. We must do this separately for each particle, to fix its individual energy. Thus in (10) we integrate over a time for each particle. Once we arrange the wave function so that each field adds a certain (almost definite) amount of both 3-momentum and energy, then we can be (almost) sure that each field only creates one particle and neither destroys particles nor creates multiple particles.

To make the particle number very definite, we need a large range in time, which is perfectly appropriate when we determine the state in terms of its particle content in the infinite past or future. Since we will need a very large range of time, we will not get a formula that is strictly of the form (10), but will involve a certain limit.

I will show a simple construction of suitable coordinate-space wave function in terms of the momentum-space wave functions $\tilde{\psi}(\mathbf{p})$ and their coordinate space counterparts $\psi(x)$. **Observe that since we use integrals over a long range of time, the coordinate space wave functions are not going to have a simple interpretation analogous to that for non-relativistic wave functions at a fixed time.**

³Note that the definition of “number of particles” is itself problematic in an interacting relativistic field theory. This merely reinforces the point. For scattering experiments we will define particle number in terms of the asymptotic in and out basis states.

4.1 FREE FIELDS

First we consider a *free* field theory, without interactions. A one-particle state with momentum-space wave function $\tilde{\psi}(\mathbf{p})$ is given by

$$\begin{aligned} |\psi \text{ free}\rangle &= \sum_{\mathbf{p}} |\mathbf{p} \text{ free}\rangle \tilde{\psi}(\mathbf{p}) \\ &= \sum_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} |0\rangle \tilde{\psi}(\mathbf{p}). \end{aligned} \quad (11)$$

The coordinate-space wave function is

$$\psi(x) = \sum_{\mathbf{p}} \tilde{\psi}(\mathbf{p}) e^{ip \cdot x}, \quad (12)$$

where $p^{\mu} = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$, so that⁴

$$|\psi \text{ free}\rangle = \int d^3\mathbf{x} \phi_{\text{free}}(x) |0\rangle 2i \frac{\partial\psi}{\partial t}. \quad (13)$$

Now consider the following state

$$|\psi, f, t_0, \Delta t\rangle = \int dt \int d^3\mathbf{x} \phi_{\text{free}}(x) |0\rangle \frac{\partial\psi}{\partial t} 2i f(t - t_0, \Delta t). \quad (14)$$

Here $f(t - t_0, \Delta t)$ is a smooth function centered at time t_0 and of width Δt . We will show how, with a suitable normalization condition on f , this state is actually the desired single-particle state. Once we understand how the derivation works, it will be easy to generalize it to multiparticle states in interacting theories. By taking a suitable limit, with $t_0 \rightarrow \pm\infty$ and $\Delta t \rightarrow \infty$, we will arrange that the field integrated with the wave function only creates a single particle, and neither destroys particles nor creates multiple particles.

It might appear natural for the definition to have a factor of ψ , as might be appropriate for creating a single particle state. However we use its time derivative $\partial\psi/\partial t$ instead, since that will provide the factor of energy needed to invert the Fourier transform over \mathbf{p} , with the relativistic normalization. Then we obtain Eq. (13) which has the field, not its time-derivative, acting on the vacuum. Then we go to Eq. (14), where the integral over time will ensure, as we will see, that we pick out the appropriate energy value as coded in the $e^{-iE_{\mathbf{p}}t}$ factor in (12).

The function f has two purposes. First by having a finite width in time, it ensures that the integral over time is absolutely convergent, which it would not otherwise be. Second, by having a certain center and width, it gives us the range of time that we are using to construct the state. When we consider the field by itself, rather than the field acting on the vacuum, and then take $\Delta t \rightarrow \infty$, it will project out the term with the creation operator $a_{\mathbf{p}}^{\dagger}$, and

⁴Note: In the standard way of organizing the proof, the time derivative is in the form $i\phi_{\text{free}}(x) \partial\psi/\partial t - i\partial\phi_{\text{free}}(x)/\partial t \psi$. This removes the annihilation operator in the free field. We are going to use an average over time to do this in a later step, to prevent the possibility of creating a multi-particle state when we use an interacting field.

eliminate the term with the destruction operator.⁵ It is worth verifying this explicitly for yourself. The same operation applied to an interacting field will project out a single-particle state uncontaminated by multi-particle components.

We now express the state in terms of momentum eigenstates:

$$\begin{aligned}
|\psi, f, t_0, \Delta t\rangle &= \int dt \sum_{\mathbf{p}, \mathbf{k}} \int d^3\mathbf{x} a_{\mathbf{p}}^\dagger |0\rangle e^{-ip \cdot x} \tilde{\psi}(\mathbf{k}) \frac{\partial e^{ik \cdot x}}{\partial t} 2i f(t - t_0, \Delta t) \\
&= \int dt \sum_{\mathbf{p}, \mathbf{k}} \int d^3\mathbf{x} a_{\mathbf{p}}^\dagger |0\rangle e^{-i(p-k) \cdot x} \tilde{\psi}(\mathbf{k}) 2E_{\mathbf{k}} f(t - t_0, \Delta t) \\
&= \int dt \sum_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle \tilde{\psi}(\mathbf{p}) f(t - t_0, \Delta t).
\end{aligned} \tag{15}$$

This equals the desired state $|\psi\rangle$ provided f obeys the condition

$$\int dt f(t - t_0, \Delta t) = 1. \tag{16}$$

A suitable function is a Gaussian

$$f(t - t_0, \Delta t) = \frac{1}{\sqrt{\pi}\Delta t} \exp\left(-\frac{(t - t_0)^2}{(\Delta t)^2}\right). \tag{17}$$

But this choice is not unique.

4.2 INTERACTING FIELDS

We now construct a formula for the initial-state $|i\rangle$ in terms of fields applied to the vacuum. To simplify the notation, define for each single particle a modified wave function:

$$\hat{\psi}_j(x; t_0, \Delta t) = \frac{\partial \psi_j}{\partial t} 2i f(x^0 - t_0, \Delta t). \tag{18}$$

This is the same factor as in (14), generalized to any single particle wave function. It is (the time derivative of) the simple one-particle wave function with the a factor f that restricts it roughly to times within a range Δt of t_0 .

Consider now the following state

$$|i; t_0, \Delta t\rangle = \iint d^4x d^4y \frac{\phi(x)}{c} \frac{\phi(y)}{c} |0\rangle \hat{\psi}_1(x; -t_0, \Delta t) \hat{\psi}_2(y; -t_0, \Delta t), \tag{19}$$

which uses the modified coordinate-space wave functions. Now the original coordinate-space wave functions $\psi_1(x)$ and $\psi_2(y)$ from which these were derived correspond to free particles. So if we take both of the times x^0 and y^0 very large and negative, these original wave functions will be localized in space at very large distances from the origin. Note that we have used

⁵If you like, you can consider this as a theorist's analog to the actual experimental set up to make beams for a scattering experiment. Real high-energy experiments involve very long paths to define beams of particles of a particular energy.

the negative of t_0 in the second argument of the modified wave function; this simply codes that we wish to analyze the state as an initial state with t_0 large and positive.

We have chosen wave functions that correspond to good wave packets, as described earlier. Now we take the limit $t_0 \rightarrow \infty$ and also let $\Delta t \rightarrow \infty$ but less rapidly: $\Delta t/t_0 \rightarrow 0$. Then the factors f ensure that the modified wave functions $\hat{\psi}_1(x; -t_0, \Delta t)$ and $\hat{\psi}_2(y; -t_0, \Delta t)$ will each be dominantly nonzero in a particular region that is centered around a classical trajectory. Most importantly the two regions are widely separated by a space-like amount; this because although the width of the range of time, Δt , gets big, the center of the range t_0 gets bigger even more quickly. See below for the utility of the space-like separation.

In the limit just defined, the state given by Eq. (19) approaches the desired initial state $|i\rangle$. To see this:

- The fact that $\Delta t \rightarrow \infty$, ensures that each field injects the amount of energy corresponding to the corresponding single-particle free-field wave function.
- The fact that the two fields $\phi(x)$ and $\phi(y)$ are being used at space-like separation means that they commute and that they independently create particles⁶. It also means we can replace the simple product of fields by their time-ordered product, which is useful, since we normally do practical calculations with Feynman rules, which apply to (vacuum expectation values of) time-ordered products of fields.
- The factors $1/c$ with each field cancel the factor c that appears in the vacuum-to-one-particle matrix element of the field (3). Thus the state has the correct normalization that corresponds to the desired state, in (6).

Our final result is that the correct formula to be used is not of the exact form (10) but is

$$|i\rangle = \lim_{\substack{t_0 \rightarrow \infty \\ \Delta t \rightarrow \infty \\ \Delta t/t_0 \rightarrow 0}} \iint d^4x d^4y \frac{\phi(x)}{c} \frac{\phi(y)}{c} |0\rangle \hat{\psi}_1(x; -t_0, \Delta t) \hat{\psi}_2(y; -t_0, \Delta t). \quad (20)$$

Similarly the final state is

$$|f\rangle = \lim_{\substack{t_0 \rightarrow \infty \\ \Delta t \rightarrow \infty \\ \Delta t/t_0 \rightarrow 0}} \int \prod_{j=1}^n d^4x_j \prod_{j=1}^n \frac{\phi(x_j)}{c} |0\rangle \prod_{j=1}^n \hat{\psi}'_j(x_j; t_0, \Delta t). \quad (21)$$

Note that in both formulae, the vacuum state $|0\rangle$ is very definitely and strictly the true vacuum. This is just the same as in the definition of the coefficient c , (3), where the vacuum and one-particle states are definitely the true vacuum and one-particle states, i.e., the true physical states. In contrast, many textbook treatments appear to suggest that the state $|0\rangle$ should be the free-field unperturbed vacuum; if that approach is tried, very delicate limits involving adiabatic switching of the interaction are called for.

⁶This is the hardest statement to prove mathematically completely. It is probably best to say that the statement is very reasonable physically and corresponds to the only rational way of creating a state that can be interpreted as two incoming particles.

5 Derivation of LSZ formula

We have now constructed the states as fields applied to the vacuum. The desired matrix element is then

$$\begin{aligned}
\langle f|i\rangle &= \lim_{\substack{t_0 \rightarrow \infty \\ \Delta t \rightarrow \infty \\ \Delta t/t_0 \rightarrow 0}} \int \prod_{j=1}^2 d^4x_j \prod_{k=1}^n d^4y_k \left[\prod_{k=1}^n \hat{\psi}'_k(y_k; t_0, \Delta t) \right]^* \prod_{j=1}^2 \hat{\psi}_j(x_j; -t_0, \Delta t) \\
&\quad \frac{1}{c^{2+n}} \langle 0 | \prod_{k=1}^n \phi(y_k) \prod_{j=1}^2 \phi(x_j) | 0 \rangle \\
&= \lim_{\substack{t_0 \rightarrow \infty \\ \Delta t \rightarrow \infty \\ \Delta t/t_0 \rightarrow 0}} \int \prod_{j=1}^2 d^4x_j \prod_{k=1}^n d^4y_k \left[\prod_{k=1}^n \hat{\psi}'_k(y_k; t_0, \Delta t) \right]^* \prod_{j=1}^2 \hat{\psi}_j(x_j; -t_0, \Delta t) \\
&\quad \frac{1}{c^{2+n}} \langle 0 | T \prod_{k=1}^n \phi(y_k) \prod_{j=1}^2 \phi(x_j) | 0 \rangle. \tag{22}
\end{aligned}$$

In the last line we have used the following properties that are valid in the limit $t_0 \rightarrow \infty$:

- all of the times y_k^0 are restricted to be greater than all of the times x_j^0 ,
- all of the points x_j are mutually space-like,
- all of the points y_k are mutually space-like,

to replace the simple product of fields by the time-ordered product.

To convert this formula to momentum space, we need the Fourier transform of the Green function:

$$\begin{aligned}
\langle 0 | T \prod_{k=1}^n \phi(y_k) \prod_{j=1}^2 \phi(x_j) | 0 \rangle &= \int \prod_{j=1}^2 \frac{d^4p_j}{(2\pi)^4} \prod_{k=1}^n \frac{d^4q_k}{(2\pi)^4} e^{-i \sum_{j=1,2} p_j \cdot x_j + i \sum_{k=1}^n q_k \cdot y_k} \\
&\quad (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{j=1}^n q_j \right) G_{n+2}(p_1, p_2, -q_1, \dots, -q_{n-1}). \tag{23}
\end{aligned}$$

We insert this into (22) and perform the over the spatial coordinates with the aid of the formulae (9) and (14). The coordinate integrals are really the same as we used in the free field case. We find

$$\begin{aligned}
\langle f|i\rangle &= \sum_{\mathbf{p}_1, \mathbf{p}_2} \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \left[\prod_{k=1}^n \tilde{\psi}'_k(q_k) \right]^* \prod_{j=1}^2 \tilde{\psi}_j(p_j) \frac{1}{c^{2+n}} \\
&\quad \times \text{special factor} \times (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{j=1}^n q_j \right) G_{n+2}(p_1, p_2, -q_1, \dots, -q_{n-1}). \tag{24}
\end{aligned}$$

The ‘‘special factor’’ is a product of

$$\int \frac{dp_j^0}{2\pi} \int dt f(t + t_0, \Delta t) 2E_{p_j} \exp[i(p_j^0 - E_{p_j})t] \tag{25}$$

for each initial-state particle, and

$$\int \frac{dq_k^0}{2\pi} \int dt f(t - t_0, \Delta t) 2E_{q_k} \exp[i(-q_k^0 + E_{q_k})t] \quad (26)$$

for each final-state particle, with the usual limits applied to t_0 and Δt . These factors give the projection to on-shell energies that appears in the second form (4) of the LSZ reduction formula. To see this, consider the form of one of these integrals:

$$\lim_{\substack{t_0 \rightarrow \infty \\ \Delta t \rightarrow \infty \\ \Delta t/t_0 \rightarrow 0}} \int \frac{dE}{2\pi} \int dt f(t + t_0, \Delta t) 2E_p \exp[i(E - E_p)t] G(E, \mathbf{p}). \quad (27)$$

The main point is that as $t_0 \rightarrow \infty$ values of t around $-t_0$ dominate, and so the exponential has very rapid oscillations as a function of E . So integrating over E gives a vanishing result wherever the rest of the integrand is analytic. So we have dominance by a very small neighborhood of the pole of the Green function, at $E = E_p$.

It is simplest just to use the explicit form for f as given in (17). Then the t integral can be performed explicitly so that (27) becomes

$$\lim \int \frac{dE}{2\pi} \exp[-(E - E_p)^2(\Delta t)^2/4] 2E_p \exp[-i(E - E_p)t_0] G(E, \mathbf{p}). \quad (28)$$

The real exponential ensures that when $\Delta t \rightarrow \infty$, the integral vanishes except for a contribution from E within a distance of order $1/\Delta t$ of E_p . Thus the use of a long range of time has enabled us to fix the energy of the particle precisely. We then have to deal with the interplay of the rapidly oscillating complex exponential and the pole factor $1/(E - E_p + i\epsilon)$ in the Green function. An appropriate deformation into the lower half complex plane gives

$$2E_p \frac{1}{i} \text{residue of pole at } E = E_p \text{ in } G(E, \mathbf{p}). \quad (29)$$

The basic argument leading to this result simply has us take $t_0 \rightarrow \infty$ and correspondingly deform E into the lower half plane. We then pick up the residues of any poles in E that are in the lower half plane. One pole is in the external propagator and corresponds to the desired particle in the state. But, in general, there may be other singularities in the Green function at other points in E ; these would correspond to particles other than the desired one. The exponential factor with Δt kills the contributions from these other singularities.

The choice that $\Delta t/t_0 \rightarrow 0$ ensures that both these mechanisms apply, because we can then deform the E contour into the lower half plane by a small distance ΔE , which satisfies $1/t_0 \gg \Delta E \gg 1/\Delta t$. The first inequality implies a suppression on the deformed contour in the originally complex exponential by $e^{-\Delta E t_0}$. The second inequality implies that the deformation is not large enough to undo the suppression caused by the $\exp[-(E - E_p)^2(\Delta t)^2/4]$ factor.

The above argument can be applied to each external line in turn, and the final result is just the second form of the LSZ formula (5). Since we only examine non-forward scattering, only the connected part of the Green function contributes.

6 Generalizations

1. The precise form of f is irrelevant if it satisfies the same general specifications.
2. The same argument can be applied to the disconnected part of the Green function to get the non-scattering term in the S-matrix.
3. The same argument can be applied in any theory with any number of kinds of particle.
4. The field used in the Green function for a particular particle can be any field that has a nonzero matrix element between the vacuum and a state of one of that kind of particle. This generally means that the field should “have quantum numbers corresponding to the particle”. (E.g., for a proton we could use an operator with two anti-upquark fields and one anti-downquark field. These would be antiquark fields, because with the definition in (3), the fields need to be those that have the correct quantum numbers to create the particle.)
5. In this context the jargon is that the field is the “interpolating field” for the particle.
6. The choice of interpolating field is not unique. E.g., ϕ^3 or $\partial\phi/\partial x^\mu$ would work in the case described above.
7. The coefficient c will depend on both the kind of particle and the field used. It is in general complex; only for one field can the phase be eliminated by convention.

For another example of an interpolating field, consider a Schrödinger field theory of an electron field and a proton field, with a Coulomb interaction. This theory has single particle states not only for electrons and protons but also for every stable energy level of hydrogen atom, and in fact for any stable ion and molecule. A possible interpolating field for s -states of hydrogen would be $\psi_e^\dagger(t, \mathbf{x})\psi_p^\dagger(t, \mathbf{x})$. By choice of different time-dependent wave functions in the derivation of the LSZ formula we can pick out different energy levels of the atom for the particle that appears in an S-matrix element. If we want to deal with all energy levels it is convenient to separate the electron and proton fields: $\psi_e^\dagger(t, \mathbf{x} + \mathbf{y})\psi_p^\dagger(t, \mathbf{x} - \mathbf{y})$.

Particularly notable applications in strong interactions are for pseudo-scalar mesons like the pion, since for the pion, certain Noether currents for symmetries can be used as interpolating fields. Interesting results can be found by applying Ward identities for the symmetries to the Green functions used in the LSZ formula. This gets into the subject known as “current algebra”.

References

- [1] H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cim.* **1**, 25 (1955).
This paper is in German, and I know of no translation into English. However, it is actually a quite clear paper. If anyone would like to read it, I have a pdf file of it.