Final subject:

Symmetries : various types:

$\boxed{\text{global}}$ ~~local~~ → 3rd quarter

ch 22 – continuous discrete – ch 23

abelian non-Abelian – ch. 24

ch. 30-32 – spontaneously broken unbroken

exact approximate.

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(optional) Makeup lecture: Friday Dec. 7 or Weds. Dec. 5
(if enough interest).

Possible subjects:

- conformal symmetry.
- nonlinear sigma models + large-N limit.
- something else.

please email me if interested, with subject(s) of interest, time conflicts on 12/5 and 12/7.

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Final exam: 24hr: when?

- conserved currents.
- Ward identities
- energy momentum tensor.
- Coleman-Mandula theorem.

Continuous symmetry.

Consider scalar fields $\phi^a(x)$, and Lagrangian density

$\mathcal{L}(x) = \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$. Equation of motion:

$$0 = \delta S = \int d^4x \; \delta\phi_a(x) \frac{\partial \mathcal{L}(x)}{\partial \phi_a(x)} + \partial_\mu \delta\phi_a(x) \frac{\partial \mathcal{L}(x)}{\partial[\partial_\mu \phi_a(x)]}$$

$$= \int d^4x \; \delta\phi_a(x) \left(\frac{\partial \mathcal{L}(x)}{\partial \phi_a(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \phi_a(x))} \right) + \text{surface term.}$$

~~Con the $\delta\phi$.~~ In order that this be true for arbitrary $\delta\phi_a(x)$, we must have

$$\boxed{\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a(x))} = \frac{\partial \mathcal{L}}{\partial \phi_a(x)}}$$

(cf. $\dot{\left(\frac{\partial L}{\partial \dot{q}}\right)} = \frac{\partial L}{\partial q}$ for $L(q, \dot{q})$)

Now, consider some specific transformation

$$\phi_a(x) \to \phi_a(x) + \varepsilon f_a(x)$$

$$\delta \mathcal{L}(x) = \underbrace{\frac{\partial \mathcal{L}(x)}{\partial \phi_a(x)} \varepsilon f_a(x)}_{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a(x))}\right] \cdot \varepsilon f_a(x)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a(x))} \varepsilon \partial_\mu f_a(x)$$

$$= \varepsilon \, \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a(x))} f_a(x) \right]}_{\equiv \mathcal{J}^\mu(x)}$$

$$\partial_\mu \mathcal{J}^\mu(x) = \frac{1}{\varepsilon} \delta \mathcal{L}(x)$$

If ~~$\phi_a(x) \to \phi_a(x) + \varepsilon f_a(x)$ is a sy~~ $\delta \mathcal{L} = 0$,

this transformation is a symmetry of the Lagrangian, and

we see that $\quad \partial_\mu j^\mu = 0$

(~~Noeth~~ Noether's theorem, simplest case): symmetry $\leftrightarrow$ conserved charge

$$\underset{\substack{\uparrow \\ \text{charge} \\ \text{dens}}}{\partial_t j^0} + \underset{\substack{\uparrow \\ \text{current}}}{\vec{\partial} \cdot \vec{j}} = 0$$

$$Q = \int d^3x \; j^0 \qquad \frac{dQ}{dt} = \int d^3x \; \partial_t j^0 = -\int d^3x \; \vec{\partial} \cdot \vec{j}$$

$= 0$ unless there is charge flowing out at ∞.

Example: $\mathcal{L} = -\partial^\mu \phi^+ \partial_\mu \phi - m^2 \phi^+ \phi - \frac{1}{4} \lambda (\phi^+ \phi)^2$

$$= -\frac{1}{2} \partial^\mu \phi_1 \partial_\mu \phi_1 - \frac{1}{2} \partial^\mu \phi_2 \partial_\mu \phi_2 - m^2 (\phi_1^2 + \phi_2^2) - \frac{1}{4} \lambda (\phi_1^2 + \phi_2^2)^2$$

$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$

Symmetry $\quad \phi \to e^{-i\theta} \phi \qquad$ Let $\theta = \varepsilon$: $\quad \delta\phi = -i\varepsilon\phi$

$\delta\phi_2 = -\varepsilon \phi_1 \quad \delta\phi_1 = +\varepsilon \phi_2 \qquad f_1 = +\phi_2 \quad f_2 = -\phi_1$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = -\partial^\mu \phi_a \quad \to \quad j^\mu = -\phi_2 \partial^\mu \phi_1 + \phi_1 \partial^\mu \phi_2$$

$$\partial_\mu j^\mu = 0 \quad \text{(check)}.$$

or, work with ϕ and ϕ^* directly:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -\partial^\mu_\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu_\mu \phi^*$$

$$f_\phi = -i\phi \qquad f_{\phi^*} = +i\phi^*$$

$$j^\mu = -i\phi^* \partial_\mu \phi + i\phi \partial_\mu \phi^* = -i\phi^* \overleftrightarrow{\partial_\mu} \phi$$

$$Q = \int d^3\vec{x}\; j^0$$

Let $\phi = \int \widetilde{dk} \left[a(\vec{k}) e^{ik\cdot x} + b^+(\vec{k}) e^{-ik\cdot x} \right]$

destroys a ϕ particle

creates a ϕ^*

$$Q = \int \widetilde{dk} \left[a^+(\vec{k})\, a(\vec{k}) - b^+(\vec{k}) b(-\vec{k}) \right]$$

$$N_\phi - N_{\phi^*}$$

$$\underbrace{Q\phi = \phi(Q-1)}_{\phi \text{ destroys one unit of charge.}} \qquad [Q, \phi] = -\phi = -i f_\phi$$

$$= -i\,\delta\phi/\varepsilon .$$

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Ward identity: consider

$$\partial_\mu \langle 0| T\, j^\mu(x)\, \underbrace{\phi(x_1)\phi(x_2)\phi^*(x_3)\;\phi^+(x_4)}_{\text{any number of fields.}} |0\rangle$$

$\partial_\mu j^\mu = 0$, but we have to be careful about t:

e.g., with just one field,

$$T\, j^\mu(x)\,\phi(x_1) = \Theta(t-t_1)\, j^\mu(x)\,\phi(x_1) + \Theta(t_1-t)\,\phi(x_1)\, j^\mu(x)$$

$$\frac{\partial}{\partial x^\mu} = \Theta(\partial_\mu j^\mu) + \delta(t-t_1)\,[j^0(x),\phi(x_1)]$$

$$\partial_t\,\Theta(t-t_1) = \delta(t-t_1)$$

$$\partial_t\,\Theta(t_1-t) = -\delta(t-t_1)$$

Now, $[\int d^3\vec{x}\, j^0(x),\ \phi(x_1)]$ ← equal to

$$= -i f_\phi = -i\frac{\delta\phi}{\varepsilon}$$

$$[j^0(x),\phi(x_1)] = -i\frac{\delta\phi(x)}{\varepsilon}\,\delta^3(\vec{x}-\vec{x}_1)$$

$$\partial_\mu\, T\, j^\mu(x)\,\phi(x_1) = -\frac{i}{\varepsilon}\,\delta\phi(x)\,\delta^4(x-x_1)$$

$$\partial_\mu\langle 0|T\, j^\mu\,\phi_{a_1}(x_1)\,\phi_{a_2}\cdots|0\rangle = \cancel{-i\sum_\lambda \delta^4(x-x_\lambda)\frac{\delta\phi_a(x)}{\varepsilon}}$$

$$-\frac{i}{\varepsilon}\sum_k \delta^4(x-x_k)\langle 0|T\,\phi_{a_1}(x_1)\cdots\underline{\delta\phi_{a_k}(x)}\cdots\phi_{a_n}(x_n)|0\rangle$$

(Mark derives in a different way).

"Ward identity" (a little technical, but good to know about)

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Our definition of a symmetry is not general enough.

Suppose that $\delta\mathcal{L} = \partial_\mu K^\mu(x)$ where K is some funct. of the fields. Then $S = \int \mathcal{L}$ is invariant.

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a(x))}\delta_a(x)\right] = \partial_\mu K^\mu$$

$$\rightarrow \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a(x))}\delta_a(x) - K^\mu \quad \text{is conserved.}$$

Important example: $\delta\phi_a(x) = \varepsilon a^\nu \partial_\nu \phi_a(x)$

for any fixed 4-vector a^ν (translation).

$$\delta\mathcal{L}(x) = \varepsilon a^\nu \partial_\nu \mathcal{L}(x)$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a(x))} a^\nu \partial_\nu \phi_a(x) - a^\nu \delta_\nu^\mu \mathcal{L}(x)$$

$$\equiv -a^\nu T_\nu{}^\mu$$

$$T_\nu{}^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a(x))}\partial_\nu\phi_a(x) + \delta_\nu{}^\mu \mathcal{L}(x).$$

stress-energy or energy-momentum tensor.

Examp: $\mathcal{L} = +\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)$

$$T_\nu{}^\mu = +\partial_\nu\phi\,\partial^\mu\phi - \delta_\nu{}^\mu\left(\tfrac{1}{2}\partial_\lambda\phi\,\partial^\lambda\phi + V(\phi)\right)$$

$T^{\mu\nu} = -T^{\nu\mu}$

$$T^{00} = \tfrac{1}{2}\dot\phi\dot\phi + \tfrac{1}{2}\vec\partial\phi\cdot\vec\partial\phi + V(\phi) = \mathcal{H} \quad \text{(energy dens)}$$

$$P^i = \int d^3x\, T^{0i} \qquad T^{0i} = -\dot\phi\,\partial^i\phi$$

$$= \int d\tilde{k}\; a^\dagger(k)\, a(k)\, k^\mu \qquad (\# \times 4\text{-momentum})$$

$$[P^\mu, \phi] = i\,\partial^\mu \phi$$

Recall infinitesimal Lorentz transform

$$x^\mu \to x^\mu + \varepsilon\, \omega^\mu{}_\lambda\, x^\lambda$$

$$\delta\phi = \varepsilon\, \omega^\mu{}_\lambda\, x^\lambda\, \partial_\mu \phi \qquad \omega^{\mu\lambda} = -\omega^{\lambda\mu}$$

j^μ is same as above with $a^\nu \to \omega^\nu{}_\lambda x^\lambda$:

$$= -\omega^\nu{}_\lambda\, x^\lambda\, T_\nu{}^\mu$$

$$\equiv +\tfrac{1}{2}\omega_{\nu\lambda}\, \mathcal{M}^{\mu\nu\lambda}$$

$$\mathcal{M}^{\mu\nu\lambda} = -x^\lambda T^{\nu\mu} + x^\nu T^{\lambda\mu}$$

$M^{\nu\lambda} = \int \mathcal{M}^{0\nu\lambda}$: generators of Lorentz group.

[Note: similar relations hold for fields with spin, though ~~with one addition~~ a little more complicated – ex. 36.4]

Coleman–Mandula theorem: we have found

$$j^\mu,\ T^{\mu\nu},\ M^{\mu\nu\lambda} \;\to\; Q,\ P^\mu,\ M^{\mu\nu}$$

Are there other possibilities, e.g. $P^{\mu\prime}$, a different conserved 4-vector?

No! (Two steps)

#1) Let $|\vec{k}, a\rangle$ = one particle of type a.

$$\langle \vec{k}, a|\, P^{\mu\prime}\, |\vec{k}, a\rangle = k^\mu q_a$$

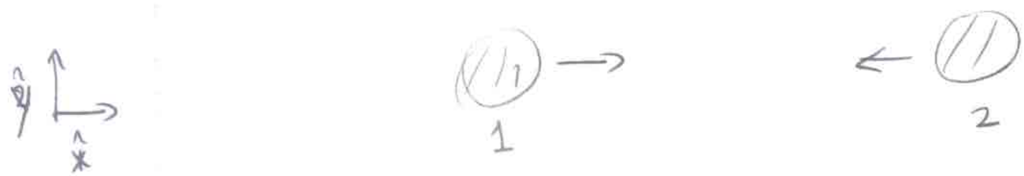
from Lorentz invar — must depend on a, or else $P^\mu \propto P^{\mu\prime}$

#2) Consider 2 particles, with $q_1 = +1$ $\quad q_2 = -1$.

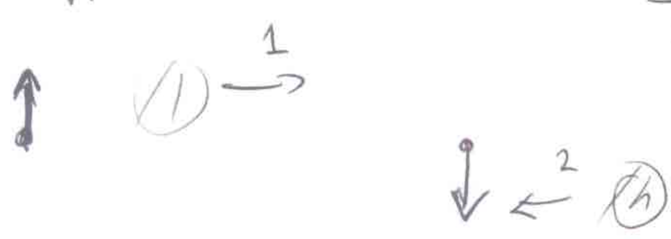
Acting on $|\vec{k}, 1\rangle$, $P^{\mu\prime} \sim +P^\mu$

on $|\vec{k}, 2\rangle$, $P^{\mu\prime} \sim -P^\mu$

Scattering



Act with $P^{\prime}$:



They miss!

$\Rightarrow$ 2 decoupled theories (trivial exception).

C+M: $M_{\mu\nu}$, P_μ, and any # of Q's.

C+M missed one possibility:

$$Q^\alpha$$

$\uparrow$ spinor index

fermionic

$\Rightarrow$ supersymmetry.

But it excludes many interesting possibilities, including some recently in the news. In particular, it severely limits symmetries that mix up spin with internal symmetries.

Weinberg, Ch. 24-B presents full details of Coleman-Mandula theorem.