Lecture 17

Discrete symmetries | Non-Abelian symmetries

Spacetime: P, T

Internal: C, Z

$P: (t, \vec{x}) \to (t, -\vec{x})$ abbreviate $x \to Px$

(~~leaves invariant~~ $-(t-t')^2 + (\vec{x}-\vec{x}')^2$, but it can't be obtained from the identity by a series of small transformations

$$P^{-1}\phi(x)P = \phi(Px) \quad \text{scalar}$$

$$P^{-1}\phi(x)P = -\phi(Px) \quad \text{pseudoscalar} \leftarrow \text{e.g. the pion}$$

$$P^{-2}\phi(x)P^{2} = \phi(x) \text{ in either case.}$$

$$S = \int d^dx \; -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \quad \text{is invariant under}$$

both, since $\partial_\mu\phi\,\partial^\mu\phi = -\dot{\phi}^2 + \underbrace{\vec{\partial}\phi\cdot\vec{\partial}\phi}_{(-1)^2}$

Is parity an automatic consequence of the other Lorentz transformations? (For fermions + gauge fields, no). Even for scalars:

~~Consider~~ Need five scalars ϕ_a.

Interaction $\quad L = \frac{2}{f}\, \varepsilon^{\mu\nu\sigma\rho}\, \phi_1 \partial_\mu \phi_2 \partial_\nu \phi_3 \partial_\sigma \phi_4 \partial_\rho \phi_5$

This is Lorentz invariant but <u>not</u> parity invariant:

$\varepsilon^{0123} = 1$, completely antisymmetric.

$\varepsilon^{\mu\nu\sigma\rho} \frac{\partial}{\partial x}$

$$L' = \varepsilon^{\mu\nu\sigma\rho}\, \phi_1 \frac{\partial}{\partial x'^\mu} \phi_2 \frac{\partial}{\partial x'^\nu} \phi_3 \frac{\partial}{\partial x'^\sigma} \phi_4 \frac{\partial}{\partial x'^\rho} \phi_5$$

$$= \underbrace{\varepsilon^{\mu\nu\sigma\rho} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\sigma} \frac{\partial x^\delta}{\partial x'^\rho}}_{} \phi_1 \partial_\alpha \phi_2 \partial_\beta \phi_3 \partial_\gamma \phi_4 \partial_\delta \phi_5$$

$$= \det \frac{\partial x}{\partial x'} \cdot \varepsilon^{\alpha\beta\gamma\delta} \ldots$$

$$= \underbrace{\det \frac{\partial x}{\partial x'}}_{} L$$

$+1$ for ordinary Lorentz transform

-1 for parity $\leftarrow$ not invariant

To be precise, this <u>is</u> invariant under

$$P^{-1} \phi_a P = -\phi_a$$

In fact, the pions + Kaons have such an interaction ($\equiv$ Wess-Zumino interaction).

To have no parity symmetry at all add also $\phi_1 \phi_2 \phi_3$ to L. (We could also consider P that reflects only <u>some</u> of the ϕ_a – if we add enough terms to L there will be no P symmetry.)

T: $\quad (t, \vec{x}) \to (-t, x) \qquad x \to Tx$

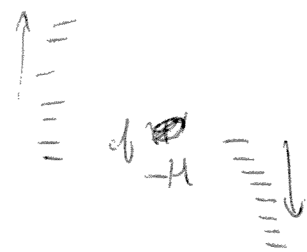
$$T^{-1}\phi(x)T = \phi(Tx)$$

$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt}$$

$$\phi(-t,\vec{x}) = e^{-iHt}\phi(0,\vec{x})e^{iHt}$$

$$T^{-1}\phi(t,x)T = T^{-1}e^{iHt}T \; T^{-1}\phi(0,\vec{x})T \; T^{-1}e^{-iHt}T$$

$$T^{-1}e^{iHt}T = e^{-iHt}$$

$T^{-1}HT \overset{?}{=} -H$: no: spectrum of H: 

Can't be conjugate!

Rather, $T^{-1}iT = -i$! : T conjugates explicit factors of i.

Again, broken ~~[illegible]~~ by the same interaction that breaks P.

special case of "CPT" theorem.

With real fields only, PT is automatically a symmetry.

With complex ϕ $\qquad C^{-1}\phi(x)C = \phi^\dagger(x)$

$$\mathcal{L} = -\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \; : \; \text{C-invariant}$$

Hermite.

Add $g\phi^4 + g^*\phi^{*4}$: if g is not real,

this breaks C. : $\to g\phi^{*4} + g^*\phi^4$

CPT is still a symmetry (T conjugates g).

and this is a theorem in QFT

- mechanical proof: all possible $\mathcal{L}$
- axiomatic proof: continue $t \to -i\tau$

rotate by π in τ-x plane

continue back.

[Capsule review: for typical scalar Lagrangians, T, P, and C are symmetries, but they need not be!]

Internal symmetry e.g. real field ϕ, $\phi \to -\phi$

(cf. $\phi \to e^{i\theta}\phi$ for a complex field).

θ: continuous parameter.

Discrete symmetries ~~don~~ don't give conserved currents, but do ~~give~~ have charges, e.g.

$Z^{-1}\phi(x)Z = \phi(-x)$.

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2\right) - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - \frac{1}{16}\lambda(\phi_1^2+\phi_2^2)^2$$

$$\phi \to \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

$$\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \to \overbrace{\begin{pmatrix}\cancel{\sin}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}}^{R}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$SO(2)$ ← 2×2

orthogonal ($R^T R = 1$)

special ($\det R = 1$)

Extend to $\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} m^2 \phi_i \phi_i - \frac{\lambda}{16}(\phi_i\phi_i)^2$

$$\phi_i(x) \to R_{ij}\,\phi_j(x)$$

independent of x (for now, global).

R_{ij}: $SO(N)$; $i = 1, \dots, N$

~~$R^T R$ is cool~~

$$\phi_i\phi_i \to R_{ij} R_{ik} \phi_j \phi_k = \phi_j \underline{R^T R} \phi$$

$SO(N)$: any R can be reached by a series of ~~infinite~~ small (near the identity) transformations. For $SO(2)$ this is familiar: $R(\alpha) = R(\alpha/m)^m$, let m be large.

The same for $SO(N)$.

Let $R_{ij} = \delta_{ij} + \varepsilon\,\Theta_{ij}$ (ε small, Θ finite)

$$R^T_{ij} R_{jk} = (\delta_{ji} + \varepsilon \Theta_{ji})(\delta_{jk} + \varepsilon \Theta_{jk})$$

$$= \delta_{ij} + \varepsilon(\underbrace{\Theta_{ki} + \Theta_{ik}}) + O(\varepsilon^2)$$

Θ_{ij} is <u>antisymmetric</u> for an infinitesimal transformation.

group: special orthogonal $N \times N$

algebra: antisymmetric $N \times N$

($\det R = 1 + \varepsilon \underbrace{\Theta_{ii}}_{=0 \text{ for antisymmetric}} + O(\varepsilon^2)$ identity)

groups: closed under multiplication (obvious: two symmetries in succession are a symmetry).

$$(R_1 R_2)^T R_1 R_2 = R_2^T \underline{R_1^T R_1}_{I} R_2 = \underbrace{\quad}_{I}$$

equivalent statement: <u>algebras</u> are closed under commutation: $[\Theta_1, \Theta_2]$ is antisymmetric if Θ_1 and Θ_2 are.

—

(ex 24.4)

Another notion: let T^a_{ij} be a complete set of antisymmetric $N \times N$ matrices. (How many? i takes N values, j takes $N-1$ because $i \neq j$, and $\times \frac{1}{2}$ for $i \leftrightarrow j$)

$\rightarrow \frac{1}{2}N(N-1)$ so $a = 1, \ldots, \frac{1}{2}N(N-1)$

Normalize $\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$ (convention for $SO(N)$)

$$[T^a, T^b] = i \underset{\substack{\uparrow \\ \text{structure constants,} \\ \text{fully antisymmetric}}}{f^{abc}} T^c \qquad (\text{closure of } [\ ,\ ])$$

$SO(2)$: only $\underline{1}$ matrix so $f^{abc} = 0$ trivially – Abelian (commuting).

$$SO(3) \quad \begin{bmatrix} & 1 & \\ -1 & & \\ & & \end{bmatrix} \quad \begin{bmatrix} & & -1 \\ & & \\ 1 & & \end{bmatrix} \quad \begin{bmatrix} & & \\ & & 1 \\ & -1 & \end{bmatrix}$$

$$f^{abc} = \varepsilon^{abc}$$

Use T^a to construct Noether currents (eq. 24.3).

⌜ Aside $O(N)$ has the same algebra → same $\underline{\text{connected}}$ part. Just add in $\underbrace{\{\phi_1 \to -\phi_1,\ \phi_i \to \phi_i \text{ for } i > 1\}}_{\det = -1} \equiv Z$

Any $R \in O(N)$ is either in $SO(N)$, or $Z \times$ element of $SO(N)$ ⌟

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu \phi_i^\dagger \partial^\mu \phi^i - m^2 \phi_i^\dagger \phi_i - \frac{\lambda}{4}(\phi_i^\dagger \phi_i)^2$$

Now: $\phi_i \to U_{ij}\phi_j$ $\qquad U =$ unitary $N \times N$

Can further separate $U = \tilde{U} \times \overbrace{e^{i\theta}}^{U(1)}$

det = 1 $\in SU(N)$

$\tilde{U} = I + i\varepsilon T$

$\tilde{U}^{+}\tilde{U} = 1 \rightarrow T^{+} = T$:

$\det \tilde{U} = 1 \longrightarrow \operatorname{Tr} T = 0$:

} hermitian traceless $N \times N$ matrices.

T^{a} = complete set $(N^2 - 1)$

$\operatorname{tr} T^{a} T^{b} = \frac{1}{2}\delta^{ab}$

different normalization from $O(N)$ — again, conventional.

$SU(2)$, $SO(3)$: same algebra. groups are 2→1:

$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$ both map to

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$