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Physics 221A **Quantum Field Theory** **Fall 2007**

http://www.kitp.ucsb.edu/~joep/Web221A/221A.html

FINAL EXAM SOLUTIONS

1. a) The propagator for field $i = A, B, C$ is $-i/(k^2 + m_i^2 - i\epsilon)$. There is an ABB vertex $-ig$ and a CBB vertex $-ih$.

b) This is identical to Srednicki Ex. 11.1, from HW#5, except that g is replaced by $g/2$. Hence (see HW solutions)

$$\Gamma = \frac{g^2\sqrt{m_A^2 - 4m_B^2}}{32\pi m_A^2} \ .$$

c) The graph is a loop of B fields with one A and two C propagators attached. An example of this type from the Standard Model is the decay of the Higgs to two gluons through a top quark loop.

d) This is identical to the calculation in Chapter 16, except that (1) we are in $d = 4$, and (2) there are two kinds of vertex, couplings g and h. Therefore, start with Eq. 16.1, with $d = 4$ and with g^3 replaced by gh^2. Carrying through to 16.8 at $\varepsilon = 2$ we have

$$\mathcal{T} = \frac{gh^2}{32\pi^2}\int dF_3\, D^{-1} \ .$$

Note that there are no factors of $\tilde{\mu}$ because the couplings are defined in $d = 4$ from the start. Also, from 16.5 we have

$$D = x_1x_3k_1^2 + x_2x_3k_2^2 + x_1x_2k_3^2 + m_B^2 = m_B^2 - x_1x_3m_A^2 - x_2(x_1 + x_3)m_C^2 \ .$$

e) In this limit $D \to m_B^2$ and we can use $\int dF_3 = 1$ to find

$$\mathcal{T} = \frac{gh^2}{32\pi^2 m_B^2} \ .$$

A coupling $\frac{1}{2}\tilde{g}AC^2$ would give $\mathcal{T} = \tilde{g}$, so

$$\tilde{g} = \frac{gh^2}{32\pi^2 m_B^2} \ .$$

The rate, from part (b), is then

$$\Gamma = \frac{\tilde{g}^2\sqrt{m_A^2 - 4m_C^2}}{32\pi m_A^2} = \frac{g^2h^4\sqrt{m_A^2 - 4m_C^2}}{2^{15}\pi^5 m_A^2 m_B^4} \ .$$

It is good to check units. The three-scalar couplings g and h have units of m^1 in $d = 4$, so Γ has units of m^1 as it should.

f) The Lagrangian has a symmetry $B \to -B$, so the number of B particles is conserved mod 2. If there is a B in the initial state, there must be at least one B in the final state.

2. a) This problem combines elements of 16.1, 28.1 and 28.3. Consider first V_4 with four external ϕ_1's. The tree level is $-Z_g g$. The relevant loops graphs are shown in the separate figure file. The three channels with ϕ_1 on the internal lines are exactly like the the case of a single ϕ field and contribute $(g^2/16\pi^2)(3/\epsilon)$ to V_4. The three channels with ϕ_2 on the internal lines each involve a λ vertex. However, the way I have defined λ, each factor of λ comes with an extra 1/6. When one has a $\phi_1^2\phi_2^2$ vertex in a graph there is a factor of 2×2 from summing over interchanging the two ϕ_1's and the two ϕ_2's; with the 1/4! in $\mathcal{L}$ the net is 1/6 (so it would have been more natural to put a 1/4 in $\mathcal{L}$). Otherwise these three graphs are the same as the others, so the net contribution to V_4 is $(\lambda^2/36 \cdot 16\pi^2)(3/\epsilon)$. Now consider the correlator with two ϕ_1's and two ϕ_2's. The tree level is $-Z_\lambda \lambda/6$ (note again the factor of 1/6), while the four graphs in the figure contribute in order $(g\lambda/6 \cdot 16\pi^2)(1/\epsilon)$, $(g\lambda/6 \cdot 16\pi^2)(1/\epsilon)$, $(\lambda^2/36 \cdot 16\pi^2)(2/\epsilon)$ $(\lambda^2/36 \cdot 16\pi^2)(2/\epsilon)$. Note in the last two the extra factor of 2, because there is no symmetry factor from interchanging the two internal lines, which are now different. Combining everything,

$$Z_g = 1 + \frac{1}{16\pi^2\epsilon}\left(3g + \frac{3\lambda^2}{36g}\right) , \qquad Z_\lambda = 1 + \frac{1}{16\pi^2\epsilon}\left(2g + \frac{4\lambda}{6}\right) .$$

The bare couplings in $d = 4 - \varepsilon$ are $g_0 = Z_\phi^{-2} Z_g g \tilde{\mu}^\varepsilon$, $\lambda_0 = Z_\phi^{-2} Z_\lambda \lambda \tilde{\mu}^\varepsilon$. The one-loop Z_ϕ vanishes as in the earlier problems. Defining $G_g = \ln Z_g$ and $G_\lambda = \ln Z_\lambda$, the conditions $\mu\partial_\mu g_0 = \mu\partial_\mu \lambda_0 = 0$ give

$$\beta_g = g(g\partial_g G_g + \lambda\partial_\lambda G_\lambda) = \frac{1}{16\pi^2}(3g^2 + \lambda^2/12) .$$

$$\beta_\lambda = \lambda(g\partial_g G_g + \lambda\partial_\lambda G_\lambda) = \frac{1}{16\pi^2}(2g\lambda + 2\lambda^2/3) .$$

b) Defining $x = \lambda/g$,

$$\mu\partial_\mu x = \frac{g\beta_\lambda - \lambda\beta_g}{g^2} = \frac{1}{16\pi^2 g^2}(-g^2\lambda + 2\lambda^2 g/3 - \lambda^3/12)$$

$$= \frac{g}{12 \cdot 16\pi^2 g^2}(-12x + 8x^2 - x^3) = -\frac{g}{12 \cdot 16\pi^2 g^2} x(x-2)(x-6) .$$

We see that there are three values of x with the property that if we start there then x doesn't change. The value $x = 0$ is easy to understand: this is the case that ϕ_1 and

ϕ_2 are completely decoupled from each other; if they start that way they stay decoupled. The value $x = 2$ corresponds to the interaction $-(g/4!)(\phi_1^2 + \phi_2^2)^2$ which has an extra $SO(2)$ symmetry, and the RG preserves this. The value $x = 6$ can also be understood: the interaction is $-(g/48)(\phi_1 + \phi_2)^4 - (g/48)(\phi_1 - \phi_2)^4$ so again there are two decoupled theories!

As we move to low energy μ decreases, so the sign of the change of x is opposite to the sign on the right-hand side of the preceding equation: positive for $0 < x < 2$ and $6 < x$, and negative for $x < 0$ and $2 < x < 6$ (see the figure page again). So if x starts anywhere in the range $0 < x < 6$ it ends up at 2. If it starts at $x > 6$ it runs toward $+\infty$. If it starts negative it runs to $-\infty$. Notice that if it starts in the range $-2 < x < 0$ the original potential is unstable, but at low energy it falls below -2: at this scale the scalars will develop a vacuum expecation value. So we have an example where the symmetry is classically unbroken but broken at low energies as a consequence of RG flow. This happens with the Higgs, in many supersymmetric models.

3. a) Notice that the λ term is the only one that depends on the relative direction of ϕ_i and χ_i, so if $\lambda < 0$ these are parallel (or antiparallel), and if $\lambda > 0$ they are perpendicular. If you just proceeded by brute force, you would find that $\partial V/\partial \phi_i$ has terms proportional to χ_i and terms proportional to ϕ_i. Therefore, either they are parallel/antiparallel, or the coefficient of χ_i (which is $\phi_j \chi_j$) vanishes, and they are perpendicular.

For $\lambda > 0$, we have two decoupled copies of the usual ϕ^4 theory, and so $\phi_i\phi_i = \chi_i\chi_i = -2m^2/g \equiv v^2$ for $m^2 < 0$. Thus ϕ_i and χ_i are any two perpendicular 3-vectors of length v.

For $\lambda < 0$ set $\chi_i = \pm\phi_i$, and the potential becomes

$$m^2\phi_i\phi_i + (g + 2\lambda)(\phi_i\phi_i)^2/4 \ ,$$

whose minimum is at $\phi_i\phi_i = -2m^2/(g + 2\lambda) \equiv v'^2$ for $m^2 < 0$. ϕ_i is any 3-vector of length v', and $\chi_i = \pm\phi_i$.

b) For $\lambda > 0$ we can rotate to

$$\phi_i = (0, 0, v) \ , \quad \chi_i = (0, v, 0) \ .$$

By itself, ϕ_i is invariant under rotations in the 1-2 plane (field space, not real space), but χ_i breaks this, so the $SO(3)$ is completely broken.

For $\lambda^2 < 0$ we can rotate to

$$\phi_i = (0, 0, v') \ , \quad \chi_i = (0, 0, \pm v') \ ,$$

so there is an unbroken symmetry of rotation in the 1-2 plane.

c) For $\lambda > 0$ write

$$\phi_i = (0, 0, v) + \phi_i' \ , \quad \chi_i = (0, v, 0) + \chi_i' \ .$$

Insert these into the potential. It is useful to write (dropping constant terms)

$$V = \frac{g}{2}(\phi_i\phi_i - v^2)^2 + \frac{g}{2}(\chi_i\chi_i - v^2)^2 + \frac{\lambda}{2}(\chi_i\phi_i)^2 \to \frac{gv^2}{8}(\phi_3'^2 + \chi_2'^2) + \frac{\lambda v^2}{2}(\chi_3' + \phi_2')^2 \ ,$$

where in the last step we have kept only quadratic terms because these are all that contribute to the masses. We see that ϕ_3', χ_2', and $\phi_2' + \chi_3'$ are massive, leaving ϕ_1', χ_1', and $\phi_2' - \chi_3'$ as the expected three massless Goldstone bosons.

For $\lambda < 0$ we have to work a bit harder. The quadratic terms are

$$V \to \frac{gv'^2}{8}(\phi_3'^2 + \chi_3'^2) + \frac{g}{4}(v'^2 - v^2)(\phi_i'\phi_i' + \chi_i'\chi_i') + \frac{\lambda v'^2}{2}(\chi_3' + \phi_3')^2 + \lambda v'^2(\phi_i'\chi_i')$$

Noting that $g(v'^2 - v^2) = -2\lambda v'^2$, the potential becomes

$$-\frac{\lambda}{2}(\phi_1 - \chi_1)^2 - \frac{\lambda}{2}(\phi_2 - \chi_2)^2 + \frac{(g + 2\lambda)v'^2}{4}(\chi_3' + \phi_3')^2 + \frac{gv'^2}{4}(\chi_3' - \phi_3')^2 \ .$$

Four linear combinations are massive, while the expected two ($\phi_1 + \chi_1$ and $\phi_2 + \chi_2$) are massless. Notice that all the masses-squared are positive in the relevant range $g > 0$, $\lambda < 0$, $g + 2\lambda > 0$.